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Solitary waves in magma dynamics

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We investigate the stability of the one-dimensional solitary waves solutions of the equations proposed by McKenzie to model the ascent of melts in the Earth interior. We show that for small porosity and two-dimensional horizontal disturbances with long wavelength, these solitary waves are unstable. We also exhibit two- and three-dimensional solitary-wave solutions of the McKenzie equations.

1. Introduction

This paper deals with certain questions which stem from the study of a mathematical model of magma flow in the Earth's mantle.

The mathematical model referred to is that due to McKenzie (1984). Building on earlier work by Walker, Stolper & Hays (1978), Ahern & Turcotte (1979) and others, McKenzie proposed a set of equations governing the motion of very slow, very viscous fluids through deformable rocks, i.e. flows relevant to many geological phenomena.

Without going into too many details, the model treats the flow of the melt through the crust essentially as a flow through a porous medium. In particular, a Darcy law is used to relate the velocity of the melt to the pressure gradient. However, the model differs from the standard porous media flows in that the pressure gradient in the melt is partly due to the deformation and compaction of the solid matrix. Thus, the medium traversed by the melt is not static, but rather fluid and solid matrix are dynamically coupled. To simplify the description of this interaction, McKenzie treats the solid matrix as a fluid with finite bulk viscosity to allow for compaction effects. The framework of two-phase fluid flows can then be used to write the dynamical equations. The reader is referred to the original paper for a complete discussion of the assumptions entering in the model. We should also mention that Scott, Stevenson & Whitehead (1986), Olson & Christensen (1986) and Whitehead (1987) have recently discussed dynamical analogs of these equations as well as laboratory experiments which are very illuminating.

In the study of the migration of melts, several authors (e.g. Richter & McKenzie 1984; Scott & Stevenson 1984; Barcilon & Richter 1986) found that a horizontal 'slab' of excess melt with a very specific vertical profile can rise under the action of buoyancy without changing shape over large distances. Said differently, McKenzie's equations admit finite amplitude, one-dimensional solitary-wave solutions. As is typical of all finite-amplitude waves, the speed of propagation of these waves is related to their maximum amplitudes. Thus, a small horizontal amplitude variation will cause a certain part of the wave to rise faster than the other. This could lead to the break-up of the wave. Such an instability has been observed in numerical calculations carried out by Scott & Stevenson (1986) as well as S. Daly & F. M. Richter (private communication).

In the present paper, we study the stability of the one-dimensional solitary waves analytically. More specifically, we consider the stability of these waves to two-dimensional disturbances with long horizontal wavelengths in the limit of small voidage (porosity). Both of these restrictions are due purely to technical reasons. The first allows us to use the modulation approach used, for example, by Kadomtsev & Petviashvili (1970) in their study of the stability of the solitary-wave solution of the Korteveg–de Vries equation. The second restriction enables us to simplify greatly the original McKenzie equations and is of interest in its own right.

Since the one-dimensional waves are unstable, the question naturally arises as to whether there are stable two- and three-dimensional ones. We investigate this question and exhibit such higher-dimension solitary waves. We also compare their properties with the one-dimensional ones.

2. The small voidage approximation

The equations proposed by McKenzie in 1984 are:

$$\phi_t + \nabla \cdot \phi \boldsymbol{v} = 0, \tag{2.1}$$

$$-\phi_t + \nabla \cdot (1-\phi)\, \boldsymbol{V} = 0, \tag{2.2}$$

$$\phi(\boldsymbol{v} - \boldsymbol{V}) + \mu^{-1} K \nabla P = 0, \tag{2.3}$$

$$-\alpha(\nabla \times \nabla \times \boldsymbol{V}) + (\eta + \tfrac{4}{3}\alpha)\nabla(\nabla \cdot \boldsymbol{V}) - \nabla P - \Delta g(1-\phi)\,\boldsymbol{k} = 0. \tag{2.4}$$

In these equations, $\boldsymbol{v}$ and $\boldsymbol{V}$ denote the velocity of the melt and solid matrix respectively; P is the dynamical part of the fluid pressure and is related to the total pressure p as follows

$$P = p + \rho_f g z,$$

where g is the gravitational acceleration, and ρ_f is the density of the melt; ϕ is the voidage, or more correctly the volumetric fraction of melt; $\boldsymbol{k}$ represents a unit vector in the vertical which is the z-direction; α and η are related to the bulk and shear viscosities α^* and η^* of the non-Stokesian fluid, which models the solid matrix, thus

$$\eta = (1-\phi)\,\eta^*,$$
$$\alpha = (1-\phi)\,\alpha^*;$$

μ is the viscosity of the melt; Δ is the difference between the solid and fluid densities, which are assumed to be constant. Finally, K is the permeability, which on the basis of laboratory data is assumed to obey the power law

$$K = K_0 \phi^3. \tag{2.5}$$

Clearly, (2.1) and (2.2) represent the conservation of mass of the melt and of the solid phases. Equation (2.3) is akin to Darcy's law, except that the pressure gradient is proportional to the velocity of the melt relative to the deformable matrix. Finally, (2.4) shows that the solid matrix deformations are treated as if they stemmed from the motion of a slow, viscous flow of fluid. McKenzie's (1984) paper discusses the assumptions which are inherent in these equations. We shall accept these equations as such and proceed with a derivation of a simpler version thereof, valid in the limit of small ϕ. To that effect, we introduce dimensionless variables as follows:

$$\left.\begin{aligned} \phi &= \phi_0 \phi', \\ K &= K^0 \phi'^3, \\ (\boldsymbol{v}, \boldsymbol{V}) &= \mu^{-1} K^0 \Delta g (\boldsymbol{v}', \boldsymbol{V}'), \\ \boldsymbol{x} &= \{\mu^{-1} K^0 (\eta + \tfrac{4}{3}\alpha)\}^{\frac{1}{2}} \boldsymbol{x}', \\ t &= \phi_0 \Delta g \{\mu(\eta + \tfrac{4}{3}\alpha)/K^0\}^{\frac{1}{2}} t', \\ P &= \Delta g \{\mu^{-1} K^0 (\eta + \tfrac{4}{3}\alpha)\}^{\frac{1}{2}} P'. \end{aligned}\right\} \tag{2.6}$$

Dropping the primes from the dimensionless variables, (2.1)–(2.4) become:

$$\phi_t + \phi_0 \nabla \cdot \phi \boldsymbol{v} = 0, \tag{2.7}$$

$$-\phi_t + \nabla \cdot (1 - \phi_0 \phi) \boldsymbol{V} = 0, \tag{2.8}$$

$$\phi_0 (\boldsymbol{v} - \boldsymbol{V}) + \phi^2 \nabla P = 0, \tag{2.9}$$

$$\{\nabla(\nabla \cdot \boldsymbol{V}) - \beta(\nabla \times \nabla \times \boldsymbol{V})\} - \nabla P - (1 - \phi_0 \phi) \boldsymbol{k} = 0, \tag{2.10}$$

where
$$\beta = \frac{\alpha}{(\eta + \frac{4}{3}\alpha)}.$$

Next, we restrict our attention to the case in which the background voidage ϕ_0 is small, and look for solutions as power series in ϕ_0 of the following form:

$$\left.\begin{aligned} \phi &= \phi^{(0)} + \phi_0 \phi^{(1)} + \dots, \\ \boldsymbol{v} &= \phi_0^{-1} (\boldsymbol{v}^{(0)} + \phi_0 \boldsymbol{v}^{(1)} + \dots) \\ \boldsymbol{V} &= \boldsymbol{V}^{(0)} + \phi_0 \boldsymbol{V}^{(1)} + \dots, \\ P &= P^{(0)} + \phi_0 P^{(1)} + \dots. \end{aligned}\right\} \tag{2.11}$$

Note the difference in the magnitudes of the melt and matrix velocities. Substituting (2.11) into (2.7)–(2.10) and dropping the zero superscripts, we obtain the following set of equations for the leading-order fields:

$$\phi_t + \nabla \cdot \phi \boldsymbol{v} = 0, \tag{2.12}$$

$$-\phi_t + \nabla \cdot \boldsymbol{V} = 0, \tag{2.13}$$

$$\phi^{-2} \boldsymbol{v} + \nabla P = 0, \tag{2.14}$$

$$\{\nabla(\nabla \cdot \boldsymbol{V}) - \beta(\nabla \times \nabla \times \boldsymbol{V})\} - \nabla P - \boldsymbol{k} = 0. \tag{2.15}$$

These simpler equations are the ones that we shall use henceforth. As a matter of fact, for infinite media, these equations can be simplified still further. Indeed, by taking the curl of (2.15) we see that

$$\nabla \times \nabla \times \nabla \times \boldsymbol{V} = 0, \tag{2.16}$$

or if we use, without loss of generality, Cartesian coordinates

$$\nabla^2 (\nabla \times \boldsymbol{V}) = 0. \tag{2.17}$$

Thus, each component of the curl of the matrix velocity is, to zeroth order, a harmonic function. Therefore, for those cases where the domain is infinite and the motion confined, then $\nabla \times V$ tends to zero at infinity, and as a result $\nabla \times V = 0$ everywhere. Physically, this means that rotational deformations of the matrix are either due to boundary effects or they are higher-order effects in ϕ_0. These assumptions hold in the present study of solitary waves. Whenever they do, the equations reduce to:

$$\phi_t + \nabla \cdot \phi \boldsymbol{v} = 0, \tag{2.18}$$

$$-\phi_t + \nabla \cdot V = 0, \tag{2.19}$$

$$\phi^{-2}\boldsymbol{v} + \nabla P = 0, \tag{2.20}$$

$$\nabla(\nabla \cdot V) - \nabla P - \boldsymbol{k} = 0, \tag{2.21}$$

$$\nabla \times V = 0. \tag{2.22}$$

If we were to eliminate P, $\boldsymbol{v}$ and V from these equations, we would obtain a single nonlinear evolution equation for ϕ:

$$\phi^3 \nabla^2 \phi_t + 3\phi^2 \nabla\phi \cdot \nabla\phi_t - \phi_t - 3\phi^2 \phi_z = 0. \tag{2.23}$$

For the one-dimensional case, this equation reduces to the one considered by Barcilon & Richter (1986). Incidentally, we note in passing that the conservation laws obtained in that paper for the one-dimensional case, can be trivially generalized as follows

$$\left.\begin{aligned} \mathscr{T}_0 &= \phi - 1, \\ X_0 &= \phi^3(\nabla\phi_t - \boldsymbol{k}), \end{aligned}\right\} \tag{2.24}$$

and

$$\left.\begin{aligned} \mathscr{T}_1 &= \nabla\phi \cdot \nabla\phi + \phi^{-1} - 1, \\ X_1 &= -\phi(\nabla\phi_t) + 3\phi\boldsymbol{k}. \end{aligned}\right\} \tag{2.25}$$

Indeed, one can check that

$$\frac{\partial \mathscr{T}_i}{\partial t} + \nabla \cdot X_i = 0,$$

for $i = 1, 2$ are identically satisfied if ϕ is a solution of (2.23).

3. Stability of one-dimensional solitary waves

The numerical work of Scott & Stevenson (1986) as well as that of S. Daly & F. M. Richter (unpublished work) shows clearly that the one-dimensional solitary waves are unstable. These early reports have prompted us to examine this question analytically. In view of the difficulties associated with such an analysis, we have restricted our attention to perturbations which (i) are two-dimensional and (ii) have long wavelengths in the horizontal direction. The ratio of the characteristic lengthscales in the vertical and horizontal directions is therefore a small parameter, say, ϵ, which we exploit to make progress.

A brief summary of the results from the one-dimensional case will prove useful. The one-dimensional solitary waves are solutions of the evolution equation (2.23) for the voidage which depends on space and time solely through the variable

$$\zeta = z - ct. \tag{3.1}$$

Therefore, if F denotes the characteristic shape of these waves, then

$$F^3 F_{\zeta\zeta\zeta} + 3F^2 F_\zeta F_{\zeta\zeta} - F_\zeta + (3/c) F^2 F_\zeta = 0, \tag{3.2}$$

with

$$F \rightarrow 1 \quad \text{as } \zeta \rightarrow \pm\infty. \tag{3.3}$$

The solution to this nonlinear differential equation is given implicitly by

$$|\zeta| = (A+\tfrac{1}{2})^{\frac{1}{2}}\left\{2(A-F)^{\frac{1}{2}} - \frac{1}{(A-1)^{\frac{1}{2}}}\ln\frac{(A-1)^{\frac{1}{2}}-(A-F)^{\frac{1}{2}}}{(A-1)^{\frac{1}{2}}+(A-F)^{\frac{1}{2}}}\right\}, \tag{3.4}$$

where A, which is always greater than 1, stands for the amplitude of the wave at the origin of the moving coordinate frame. Equation (3.4) shows that F is a function of two variables, namely ζ and A. The all-important relation between phase speed and amplitude is obtained by considering the second integral of (3.3)–(3.4),

$$F_\zeta^2 = \frac{(F-1)^2}{cF^2}(c-1-2F), \tag{3.5}$$

and evaluating it at the origin. It implies that:

$$c = 2A+1. \tag{3.6}$$

Therefore, we can equally well look upon F as a function of ζ and c and write (3.4) as

$$F = F(\zeta, c). \tag{3.7}$$

In studying the stability of these one-dimensional waves, we shall not follow the traditional procedure which consists in examining the evolution of an infinitesimal two-dimensional perturbation. Rather, we shall adopt an approach akin to that used by Kadomtsev & Petviashvili (1970) in their study of the stability of solitary waves associated with the Korteweg–de Vries equation. Essentially, this approach consists of looking for two-dimensional solutions of (2.23) which are, so to speak, 'near' the one-dimensional solitary-wave solution. To that effect we try the Ansatz

$$\phi = \psi(\eta, Y, T, \epsilon), \tag{3.8}$$

where

$$\left.\begin{aligned} \eta &= z-\Theta/\epsilon, \\ Y &= \epsilon y, \\ T &= \epsilon t. \end{aligned}\right\} \tag{3.9}$$

The variables Y and T represent slow spatial and temporal variables: Y is used to introduce a long horizontal wavelength modulation; T is used to follow the slow evolutions of the small departures from the one-dimensional wave. The justification for this slow time variable will appear in our subsequent analysis. The coordinate η is reminiscent of ζ, except that Θ is now a function of the slow variables. We therefore allow different parts of the wave to travel at different speeds. Because of the smallness of ϵ, we represent this unknown phase function as asymptotic series of the form

$$\Theta(Y,T,\epsilon) = \Theta^{(0)}(Y,T)+\epsilon\Theta^{(1)}(Y,T)+\dots. \tag{3.10}$$

Similarly, ψ is expanded as

$$\psi(\eta,Y,T,\epsilon) = \psi^{(0)}(\eta,Y,T)+\epsilon\psi^{(1)}(\eta,Y,T)+\dots. \tag{3.11}$$

Before we proceed with the stability analysis, we should warn the reader that strictly speaking there are no solutions 'nearby' a solitary wave of the form (3.8). Or rather, 'nearby' solutions are of this form only for $|\eta| \ll \epsilon^{-1}$. We shall return to this point later.

The zeroth-order approximation of (2.23) now reads

$$\Theta_T^{(0)}\{(1+(\Theta_Y^{(0)})^2)(\psi^{(0)3}\psi_{\eta\eta\eta}^{(0)}+3\psi^{(0)2}\psi_\eta^{(0)}\psi_{\eta\eta}^{(0)})-\psi_\eta^{(0)}\}+3\psi^{(0)2}\psi_\eta^{(0)} = 0. \tag{3.12}$$

We can now be more precise about what we meant by looking for a nearby solution. We want the zeroth-order approximation to be identical to the one-dimensional solitary wave. We therefore require that

$$\Theta_Y^{(0)} = 0. \tag{3.13}$$

In fact, let us redefine the phase function and write

$$\Theta_T = c^{(0)}(T) + \epsilon\theta_T, \tag{3.14a}$$

$$\Theta_Y = \epsilon\theta_Y, \tag{3.14b}$$

where
$$\theta = \theta^{(0)}(Y,T) + \epsilon\theta^{(1)}(Y,T) + \dots. \tag{3.15}$$

Equation (3.12) is now identical to (3.2) for F. As a result

$$\psi^{(0)}(\eta, Y, T) = F(\eta, c^{(0)}(T)). \tag{3.16}$$

Before turning our attention to the first-order correction to the evolution equation, we should note that:

$$\left.\begin{aligned}
\phi_z &= \psi_\eta,\\
\phi_t &= -c^{(0)}\psi_\eta + \epsilon(\psi_T - \theta_T\psi_\eta),\\
\phi_y\phi_{yt} &= -\epsilon^2 c^{(0)}(\psi_{\eta Y} - \theta_Y\psi_{\eta\eta})(\psi_Y - \theta_Y\psi_\eta) + O(\epsilon^3),\\
\phi_{yyt} &= -\epsilon^2 c^{(0)}\{(\theta_Y)^2\psi_{\eta\eta\eta} - 2\theta_Y\psi_{\eta\eta Y} - \theta_{YY}\psi_{\eta\eta} + \psi_{\eta YY}\} + O(\epsilon^3).
\end{aligned}\right\} \tag{3.17}$$

As a result, the first-order equation is

$$L\psi^{(1)} = (F^3F_{\eta\eta T} + 3F^2F_\eta F_{\eta T} - F_T) + \frac{3\theta_T^{(0)}}{c^{(0)}}F^2F_\eta, \tag{3.18}$$

where
$$L \equiv \frac{\partial}{\partial\eta}\left[c^{(0)}\left(F^3\frac{\partial^2}{\partial\eta^2} + 3F^2F_{\eta\eta} - 1\right) + 3F^2\right]. \tag{3.19}$$

The solution $\psi^{(1)}$ should also satisfy the following conditions at infinity

$$\psi^{(1)} \to 0 \quad \text{as } \eta \to \pm\infty. \tag{3.20}$$

Note that the effects of the horizontal variation are not yet felt. At this order, the stability analysis is therefore similar to that of a one-dimensional solitary wave perturbed by a one-dimensional disturbance. This is another simplification due to the choice of long wavelength disturbances.

The problem (3.18)–(3.20) for $\psi^{(1)}$ is very similar to one studied by Kodama & Ablowitz (1981). They were interested in the slow evolution of solitons of the Korteweg–de Vries equation (KdV), the modified KdV equation, the nonlinear Schrödinger equation, etc. as they travel through a slightly inhomogeneous media. Their analysis, as well as that of others (Kaup & Newell 1978; Johnson 1973; Knickerbocker & Newell 1980), shows that the solution is not uniformly valid in η. We shall have to keep this in mind as we proceed with the determination of $c^{(0)}$, $\theta^{(0)}$, etc. by means of the Fredholm alternative (see e.g. Friedman 1956, p. 45). To that effect, we find all the solutions of

$$L^*f = 0, \tag{3.21}$$

where
$$L^* = -\left[c^{(0)}\left\{\frac{\partial^2}{\partial\eta^2}F^3 + 3F^2F_{\eta\eta} - 1\right\} + 3F^2\right]\frac{\partial}{\partial\eta}, \tag{3.22}$$

is the adjoint of L. From the very definition of L^*, these solutions f must be bounded and tend to the same value at both plus and minus infinity. Clearly, $f_1 = 1$ is one such solution. Therefore, if we integrate (3.18) over η from $-\infty$ to $+\infty$, we are left after a little algebra with:

$$\frac{\partial}{\partial T}\langle F-1\rangle = 0, \tag{3.23}$$

where

$$\langle\,\rangle = \int_{-\infty}^{+\infty} \mathrm{d}\eta.$$

In the Appendix we derive an expression for the first 'conserved density' $\langle F-1\rangle$ and show that

$$\langle F-1\rangle = \tfrac{2}{3}c^{\frac{3}{2}}(c-3)^{\frac{1}{2}},$$

where c stands for $c^{(0)}$. Therefore (3.23) implies that

$$\frac{\partial c^{(0)}}{\partial T} = 0. \tag{3.24}$$

A consideration of a second linearly independent solution of the adjoint operator, namely $f_2 = F^{-2}$, would lead to the requirement that $\langle F_\eta^2 + F^{-1} - 1\rangle$, i.e. the second 'conserved density', be independent of T. This condition is automatically satisfied if (3.24) holds. Finally, the third independent solution of (3.21) is unbounded and hence need not be considered.

Thus to leading order, the phase speed of the perturbed solitary wave is the same constant as in the strict one-dimensional case.

We return to (3.18) which we now write thus:

$$L\psi^{(1)} = \frac{3\theta_T^{(0)}}{c^{(0)}}F^2F_\eta. \tag{3.25}$$

The most general bounded solution of (3.25) is

$$\psi^{(1)} = DF_\eta + \theta_T^{(0)}F_c. \tag{3.26}$$

Here, D is a function of Y and T multiplying the solution of the homogeneous part of the equation. That $\theta_T^{(0)}F_c$ is a particular solution of (3.26) can be seen by differentiating (3.2) with respect to c.

The form of $\psi^{(1)}$ is similar to the order ϵ correction of a true one-dimensional wave with speed $c^{(0)} + \epsilon c^{(1)}$. Indeed,

$$F(z-(c^{(0)}+\epsilon c^{(1)})t, c^{(0)}+\epsilon c^{(1)}) = F(\zeta, c^{(0)}) + \epsilon\{-c^{(1)}tF_\zeta + c^{(1)}F_c\} + O(\epsilon^2), \tag{3.27}$$

where ζ now stands for $z - c^{(0)}t$. This similarity between the above ϵ correction and $\psi^{(1)}$ is due to the fact that the horizontal variations are not felt to this order, a fact already alluded to. Incidentally, (3.27) also shows that the series for the one-dimensional wave will become disordered after a long time $t = O(\epsilon^{-1})$. This is the promised justification for introducing T in our analysis. Also, because we have introduced the coordinate η containing the variable phase $\Theta(Y, T, \epsilon)$ to eliminate the possibility of such a disorder, we can eliminate the homogeneous solution F_η of (3.25) by setting

$$D = 0. \tag{3.28}$$

Consequently:

$$\psi^{(1)} = \theta_T^{(0)}F_c. \tag{3.29}$$

The second-order equation, which involves the horizontal variations for the first time, reads:

$$L\psi^{(2)} = \theta^{(0)}_{TT}[F^3F_{c\eta\eta}+3F^2F_\eta F_{c\eta}-F_c]+\frac{3}{c^{(0)}}\theta^{(1)}_T F^2F_\eta$$
$$-c^{(0)}(\theta^{(0)}_Y)^2(F^3F_{\eta\eta\eta}+3F^2F_\eta F_{\eta\eta})+\tfrac{1}{2}\theta^{(0)2}_T LF_{cc}+ + c^{(0)}\theta^{(0)}_{YY}F^3F_{\eta\eta}. \quad (3.30)$$

In arriving at this equation, we have used the fact that

$$LF_{cc}+L_cF_c = -\frac{3}{c^{(0)2}}F^2F_\eta+\frac{6}{c^{(0)}}FF_\eta F_c+\frac{3}{c^{(0)}}F^2F_{c\eta}$$

to write various combinations of terms in the right-hand side in a compact form. Once again we appeal to the Fredholm alternative to determine the unknown phase. Multiplying by $f_1 = 1$ and integrating over η, we see that:

$$0 = -\theta^{(0)}_{TT}\frac{\partial}{\partial c}\langle F-1\rangle - 3c^{(0)}\theta^{(0)}_{YY}\langle F^2F^2_\eta\rangle. \quad (3.31)$$

Had we used $f_2 = F^{-2}$, we would have found that

$$0 = -\theta^{(0)}_{TT}\frac{\partial}{\partial c}\left\langle F^2_\eta+\frac{1}{F}-1\right\rangle + c^{(0)}\theta^{(0)}_{YY}\langle F^2_\eta\rangle. \quad (3.32)$$

Since these two equations for θ^0 are incompatible, we have reached the conclusion that there are no 'nearby' solutions of the form (3.8). This conclusion is not surprising in the light of the work by Johnson (1973), Kaup & Newell (1978), Kodama & Ablowitz (1981) and others already alluded to. Indeed, these authors have shown that as solitons for the Korteweg–de Vries equation (KdV), the modified KdV equation, the nonlinear Schrödinger equation, etc. travel through slightly inhomogeneous media, they evolve into waves which are no longer invariant in a moving frame. In particular, far from the humps, i.e. in the far field, fore and aft asymmetries occur.

In order to avoid an analysis of the far field, we shall assume that the instability develops in the vicinity of the hump. We compensate for the fact that our Ansatz is not valid in the far field by using that solution of the adjoint problem which tends to zero at infinity, viz.

$$f_3 = 1-F^{-2}. \quad (3.33)$$

The desired evolution equation is

$$0 = \theta^{(0)}_{TT}\frac{\partial}{\partial c}\left\langle F-1+F^2_\eta+\frac{1}{F}-1\right\rangle + c^{(0)}\theta^{(0)}_{YY}\langle(3F^2-1)F^2_\eta\rangle. \quad (3.34)$$

The computations carried out in the Appendix show that the coefficient of $\theta^{(0)}_{TT}$ is positive. Since that of $\theta^{(0)}_{YY}$ is obviously also positive, the above equations admit exponentially growing solutions. Hence both the ϵ correction to the speed and amplitude grow exponentially over the slow timescale.

This analysis proves that one-dimensional solitary waves are unstable to horizontal perturbations.

4. Multidimensional solitary waves

We are not able to follow the evolution of the unstable one-dimensional waves to their final form as two- or three-dimensional entities. Instead, we start *ab initio* to look for two- and three-dimensional waves propagating in the z-direction without changing shape. For this purpose, we introduce once again a frame of reference moving with the wave and recall that

$$\zeta = z - ct.$$

By definition, in this moving frame the form of the waves are 'permanent' and hence the governing equation is time independent. Therefore, (2.23) becomes:

$$-c\phi^3\nabla^2\phi_\zeta - 3c\phi^2\nabla\phi\cdot\nabla\phi_\zeta + c\phi_\zeta - 3\phi^2\phi_\zeta = 0, \tag{4.1}$$

where
$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial \zeta}. \tag{4.2}$$

We look for solutions of (4.2) such that

$$\phi - 1, \quad |\nabla\phi|, \quad |\nabla\phi_\zeta| \in L_2(\boldsymbol{R}). \tag{4.3}$$

We also restrict ourselves to positive solutions. Aside from the fact that a negative porosity is meaningless physically, solutions of (4.1) in which ϕ vanishes on a surface present problems. Indeed, for these solutions all the derivatives of ϕ also vanish on that surface and, as is typical of cases where there is no Lipschitz continuity, the solution is not unique. In particular, the solutions on either sides of the surface decouple. For all these reasons, we shall assume that

$$\phi > 0. \tag{4.4}$$

Under those conditions, we can prove a general result about the sign of the speed of the solitary waves, namely we can show that they must propagate upward. Since these waves are excess melt waves, they represent regions that are more buoyant than their surroundings: in that sense this result is hardly surprising.

THEOREM 1. *If a solution of* (4.1)–(4.4) *exists, then* $c \geqslant 0$.

Proof. We multiply (4.1) by ϕ_ζ and integrate over $\boldsymbol{R}$

$$-c\int_{\boldsymbol{R}} \phi_\zeta \nabla\cdot(\phi^3\nabla\phi_\zeta)\,\mathrm{d}V + c\int_{\boldsymbol{R}} |\phi_\zeta|^2\,\mathrm{d}V = 3\int_{\boldsymbol{R}} |\phi\phi_\zeta|^2\,\mathrm{d}V.$$

and after an integration by parts

$$c = \frac{3\int_{\boldsymbol{R}} |\phi\phi_\zeta|^2\,\mathrm{d}V}{\int_{\boldsymbol{R}} (\phi^3|\nabla\phi_\zeta|^2 + |\phi_\zeta|^2)\,\mathrm{d}V}, \tag{4.5}$$

which, on account of (4.4), shows that c is positive.

If we define
$$\|\phi\|_\infty = \sup_{\boldsymbol{R}} |\phi|,$$

then clearly (4.5) implies that $\quad c \leqslant 3\|\phi\|_\infty^2.$

Consequently, if $c \geqslant 3$, then the voidage must exceed 1 somewhere in the medium. We shall show that for axially and spherically symmetric waves, the voidage is in fact everywhere greater than 1 for these values of c. That this is not true for $c \leqslant 3$ can be seen from the far-field behaviour of ϕ. To that effect, we replace ϕ by 1 in (4.1) to get an equation for the leading-order term in a far-field approximation, say Φ:

$$-c\nabla^2\Phi_\zeta + (c-3)\,\Phi_\zeta = 0. \tag{4.6}$$

In two and three dimensions, ϕ oscillates in the far field about 1 whenever c is smaller than 3. For this range of phase speeds, the decay of $\phi - 1$ is not sufficient to ensure integrability of conserved quantities (2.24)–(2.25). As such, $c \leqslant 3$ is not a physically meaningful range for solitary waves. Thus, in view of (4.3), we restrict our attention to values of c greater than 3.

We next consider isotropic two- and three-dimensional waves, namely waves for which the voidage is solely a function of the single variable $x^2+\zeta^2$ and $x^2+y^2+\zeta^2$ respectively. If we define

$$\rho^2 = x^2+\zeta^2, \tag{4.7}$$

then (4.1)–(4.3) becomes

$$\phi^3\left(\phi_{\rho\rho\rho}+\frac{1}{\rho}\phi_{\rho\rho}-\frac{1}{\rho^2}\phi_\rho\right)+3\phi^2\phi_\rho\phi_{\rho\rho}-\phi_\rho+\frac{3}{c}\phi^2\phi_\rho = 0, \tag{4.8}$$

with

$$\phi-1, \quad \phi_\rho, \quad \phi_{\rho\rho}\in L_2(0,\infty). \tag{4.9}$$

Similarly, for the three-dimensional case we define without possible confusion

$$\rho^2 = x^2+y^2+\zeta^2. \tag{4.10}$$

The problem now becomes

$$\phi^3\left(\phi_{\rho\rho\rho}+\frac{2}{\rho}\phi_{\rho\rho}-\frac{2}{\rho^2}\phi_\rho\right)+3\phi^2\phi_\rho\phi_{\rho\rho}-\phi_\rho+\frac{3}{c}\phi^2\phi_\rho = 0, \tag{4.11}$$

with

$$\phi-1, \quad \phi_\rho, \quad \phi_{\rho\rho}\in L_2(0,\infty). \tag{4.12}$$

These two problems corresponding to cylindrical and spherical domains of excess melt rising through the solid matrix are so similar mathematically that we shall deal only with one of them, namely (4.8)–(4.9). All the results obtained for the cylindrical wave can easily be generalized to the spherical one.

A first and second integral of (4.8) will be needed in the sequel. A straightforward integration yields

$$\phi^3\left(\phi_{\rho\rho}+\frac{1}{\rho}\phi_\rho\right)+3\int_\rho^\infty \frac{1}{r}\phi^2\phi_r^2\,\mathrm{d}r = \frac{1}{c}(1-\phi^3)-(1-\phi). \tag{4.13}$$

If we multiply (4.13) by ϕ_ρ, divide by ϕ^2 and integrate we get after interchanging two integrations:

$$\phi_\rho^2+\int_\rho^\infty \phi_r^2(r)\left\{1-3\frac{\phi^2(r)}{\phi^2(\rho)}\right\}\frac{\mathrm{d}r}{r} = \frac{(\phi-1)^2}{\phi^2}\left\{\frac{c-(2\phi+1)}{c}\right\}. \tag{4.14}$$

Another useful integral is obtained by dividing (4.8) by ϕ^2 and integrating, namely:

$$\phi\phi_{\rho\rho}+\phi_\rho^2+\frac{1}{\rho}\phi\phi_\rho+\int_\rho^\infty \frac{1}{r}\phi_r^2\,\mathrm{d}r+(\phi-1)\left\{\frac{3}{c}-\frac{1}{\phi}\right\} = 0. \tag{4.15}$$

We are now able to prove that the two-dimensional wave has the familiar one-hump shape. We first establish

THEOREM 2. *For $c \geqslant 3$, if the solution of* (4.1)–(4.3) *exists, then $\phi \geqslant 1$.*

Proof. Let us assume that ϕ can be smaller than 1. Then ϕ has at least one minimum at say, $\rho = \rho^*$. At that point

$$\phi(\rho^*) < 1,$$

$$\phi_\rho(\rho^*) = 0,$$

$$\phi_{\rho\rho}(\rho^*) > 0.$$

Also, in view of (4.4)

$$\phi(\rho^*) > 0.$$

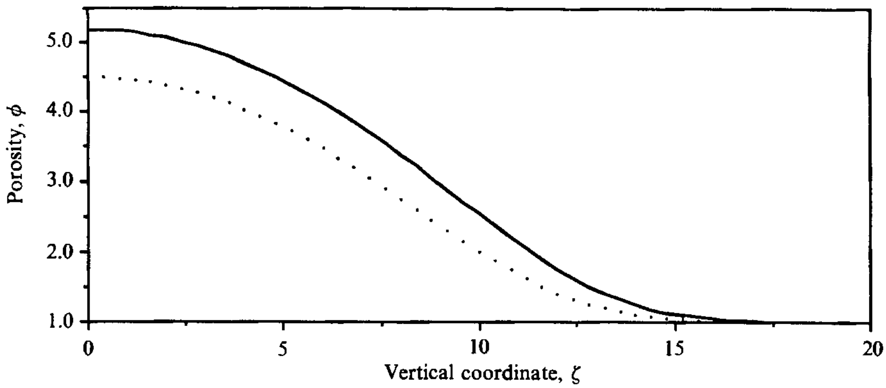

FIGURE 1. Profiles of one-dimensional (dotted line) and two-dimensional (solid line) waves for $c = 10$.

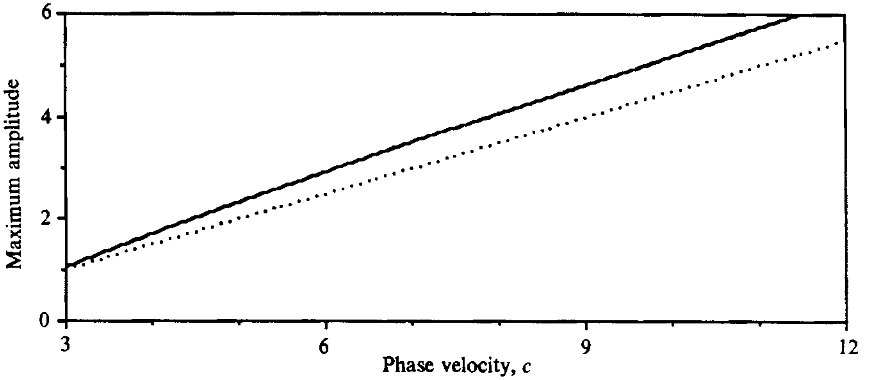

FIGURE 2. Amplitude A *vs.* phase speed c for one-dimensional (dotted line) and two-dimensional (solid line) waves.

Therefore, if we evaluate (4.15) at ρ^* we see that each term is non-negative. This is of course impossible and we are forced to conclude that $\phi \geqslant 1$.

THEOREM 3. *For cylindrical solitary waves with phase speeds $c \geqslant 3$, ϕ is a monotonic non-increasing function of the distance ρ.*

Proof. Once again we show that the assumption that the shape is not monotonic non-increasing leads to a contradiction. Indeed, if it were true, then ϕ would have at least one local minimum say at ρ_1. The value of ϕ at this point, say ϕ_1, is perforce greater than 1 since $c \geqslant 3$. And since $\phi \rightarrow 1$ as $\rho \rightarrow \infty$, there is another point, say $\rho_2 > \rho_1$, where ϕ takes on the same value. Furthermore

$$\phi(\rho) \geqslant \phi_1 \quad \text{for } \rho \in (\rho_1, \rho_2).$$

Therefore, if we evaluate (4.14) at both ρ_1 and ρ_2 and subtract we see that:

$$-\phi^2(\rho_2) + \int_{\rho_1}^{\rho_2} \phi_r^2(r)\left\{1 - 3\frac{\phi^2(r)}{\phi_1^2}\right\}\frac{\mathrm{d}r}{r} = 0,$$

which is impossible since the left-hand side is negative definite.

The actual shape of these solitary waves is not unlike that of the one-dimensional one. Figure 1, on which we have plotted the profiles for both the one- and two-dimensional waves for $c = 10$, shows that the amplitude of the two-dimensional wave

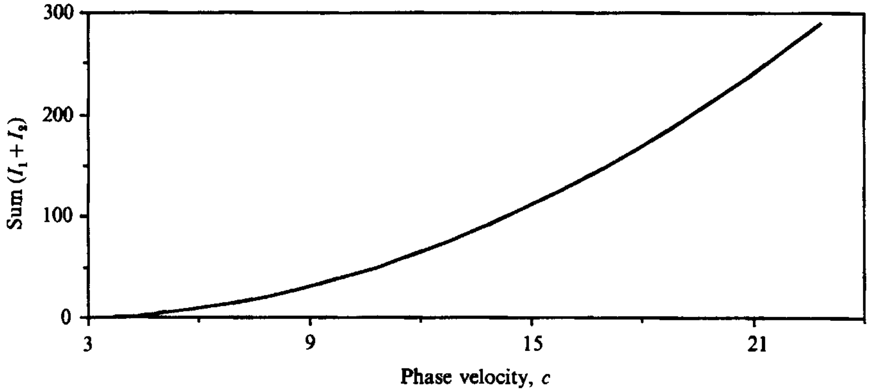

FIGURE 3. Plot of I_1+I_2 *vs.* c.

is everywhere greater than that of the one-dimensional one. Figure 2 shows a comparison between the amplitude *vs.* phase speed relation for both of these waves. At equal amplitude, the two-dimensional wave is slower than the one-dimensional one.

5. Concluding remarks

We have seen that the one-dimensional waves are unstable to two-dimensional perturbations. Most likely, this will also be the case for the two-dimensional waves we have discussed. Indeed, a local increase in amplitude will result in a faster local phase speed tearing apart the rising 'tube' of excess melt. It is therefore tempting to speculate that all the melt migration takes place by means of the three-dimensional waves.

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Appendix

In this Appendix we evaluate the one-dimensional version of the two conserved quantities since they arise in the stability analysis

$$I_1 \equiv \langle F-1 \rangle = 2\int_{-\infty}^{0} F-1\,\mathrm{d}\zeta,$$

or using F as the variable of integration

$$= 2\int_{1}^{A} (F-1)\frac{1}{F_\zeta}\mathrm{d}F$$

and in view of (3.5), $$= (2c)^{\frac{1}{2}} \int_1^A \frac{F\,\mathrm{d}F}{(A-F)^{\frac{1}{2}}}$$

or in terms of c only $$= \tfrac{2}{3}c^{\frac{3}{2}}(c-3)^{\frac{1}{2}}.$$

Similarly,

$$I_2 \equiv \left\langle F_\xi^2 + \frac{1}{F} - 1 \right\rangle = 2\int_1^A \left\{ \left(\frac{2}{c}\right)^{\frac{1}{2}} \frac{F-1}{F}(A-F)^{\frac{1}{2}} - (\tfrac{1}{2}c)^{\frac{1}{2}} \frac{1}{(A-F)^{\frac{1}{2}}} \right\} \mathrm{d}F$$

$$= -\frac{4}{3}\left(\frac{c-3}{c}\right)^{\frac{1}{2}}(c-\tfrac{3}{2}) - 4\left(\frac{c-1}{c}\right)^{\frac{1}{2}} \tanh^{-1}\left(\frac{c-3}{c-1}\right)^{\frac{1}{2}}.$$

A plot of $I_1 + I_2$ as a function of c for $c > 3$ (see figure 3), reveals that this function is monotonic increasing.

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